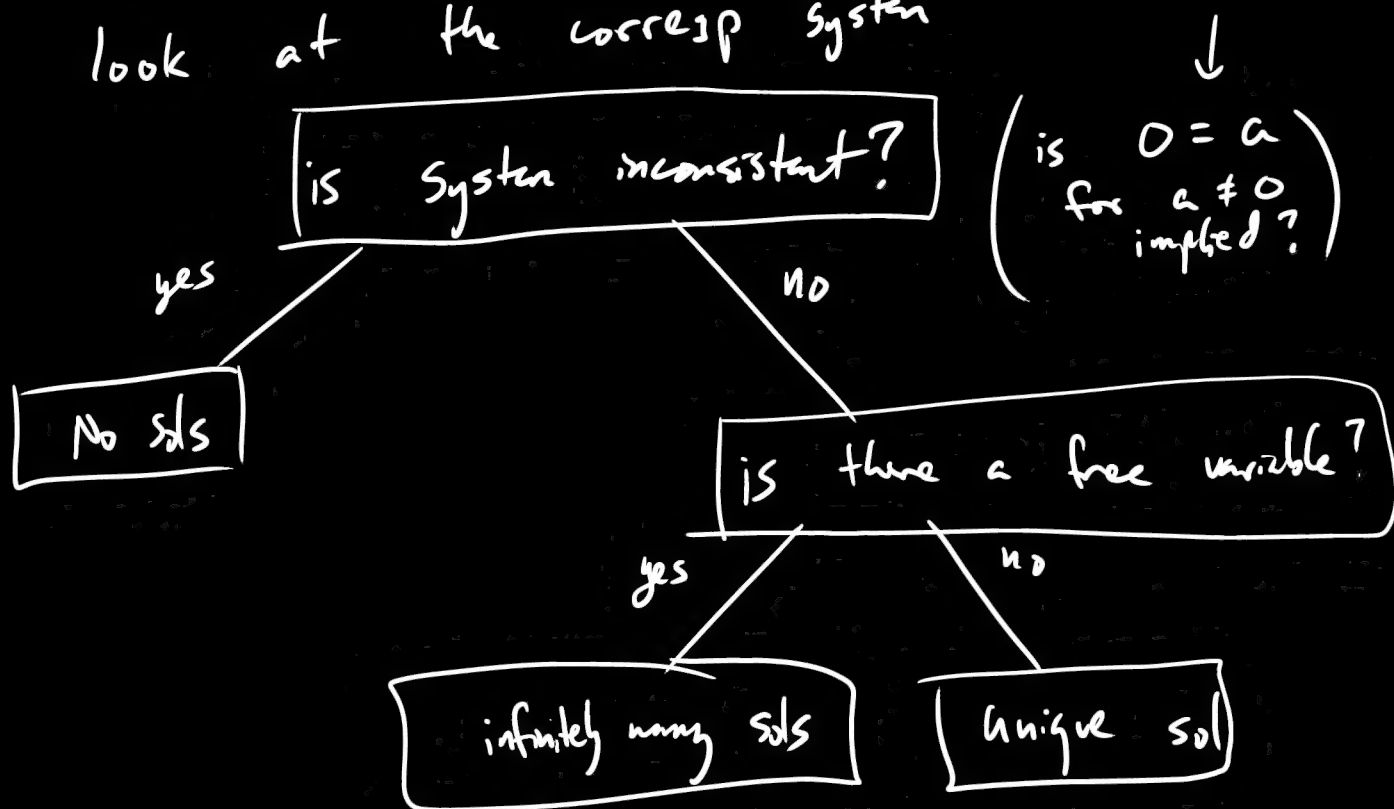


Trick (for number of solutions):

Take RREF $\rightarrow [M|b]$

look at the corresp system



Last time: RREF and Consequences...

\rightarrow briefly defined and gave examples of
linear maps / linear functions / linear homomorphisms

Refresher on Functions.

Defⁿ: A function $f: S \rightarrow T$ is a rule of assignment, i.e. a method of assigning to each element of set S a unique member of set T .

Set = collection of objects

object in the set = element = member

The domain of $f: S \rightarrow T$ is denoted

$\text{dom}(f) = S$. The codomain of f is $\text{cod}(f) = T$.

Ex: Calculus 1 is all about functions of the form $f: \mathbb{R} \rightarrow \mathbb{R}$.

ex: $\underline{f(x) = x^2}$ w/ $\text{dom}(f) = \mathbb{R}$ and $\text{cod}(f) = \mathbb{R}$.

ex: $g(x) = x^2$ w/ $\text{dom}(g) = \mathbb{R}$ and $\text{cod}(g) = \mathbb{R}_{\geq 0}$.

Ex: $L: \mathbb{R}^2 \rightarrow \mathbb{R}$ w/ $L\begin{bmatrix} x \\ y \end{bmatrix} = x + y$.

has domain \mathbb{R}^2 and codomain \mathbb{R} .

Non-Ex: "Food eaten today": People \rightarrow foods

is not a function, even though it is a rule of assignment (non-unique outputs)...

Non-Ex: $y = \pm\sqrt{1-x^2}$ describes a circle in \mathbb{R}^2 ,

★ but it is NOT a function because some input x (e.g. $x=0$) has two associated output values.

Defⁿ: A linear map is a function $L: \mathbb{R}^n \rightarrow \mathbb{R}^m$ satisfying for all $\vec{x}, \vec{y} \in \mathbb{R}^n$ and all $a \in \mathbb{R}$

$$\textcircled{1} L(\vec{x} + \vec{y}) = L(\vec{x}) + L(\vec{y}) \quad \textcircled{2} L(a\vec{x}) = aL(\vec{x}).$$

NB: the definition from last time is equivalent to this one (i.e. any map satisfying that condition satisfies the new one and vice versa).

Prop: Suppose $L: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a function. The following are equivalent:

- ① for all $\vec{x}, \vec{y} \in \mathbb{R}^n$ and all $a \in \mathbb{R}$ we have both
 $\rightarrow L(\vec{x} + \vec{y}) = L(\vec{x}) + L(\vec{y})$ and $L(a\vec{x}) = aL(\vec{x})$.
- ② for all $\vec{x}, \vec{y} \in \mathbb{R}^n$ and all $a \in \mathbb{R}$ we have
 $L(\vec{x} + a\vec{y}) = L(\vec{x}) + aL(\vec{y})$.

Len: Linear maps in either sense always map the zero vector to the zero vector.

pf(Len): Let $L: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a function.

- ① Assume $L(\vec{x} + \vec{y}) = L(\vec{x}) + L(\vec{y})$ and $L(a\vec{x}) = aL(\vec{x})$ for all $\vec{x}, \vec{y} \in \mathbb{R}^n$ and all $a \in \mathbb{R}$.

$$\text{Then } L(\vec{0}) = L(0 \cdot \vec{0}) = 0 L(\vec{0}) = \vec{0}.$$

- ② Assume $L(\vec{x} + a\vec{y}) = L(\vec{x}) + aL(\vec{y})$ for all $\vec{x}, \vec{y} \in \mathbb{R}^n$ and $a \in \mathbb{R}$.

$$L(\vec{0}) = L(\vec{0} + (-1) \cdot \vec{0}) = L(\vec{0}) - 1L(\vec{0}) = \vec{0}.$$

Hence $L(\vec{0}) = \vec{0}$ in either case. □

pf (of Proposition): Let $L: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a function.

① \Rightarrow ②: Assume $L(\vec{x} + \vec{y}) = L(\vec{x}) + L(\vec{y})$ ✓
 $L(a\vec{x}) = aL(\vec{x})$ for all $\vec{x}, \vec{y} \in \mathbb{R}^n$ and all $a \in \mathbb{R}$.

Thus $L(\vec{x} + a\vec{y}) = L(\vec{x} + (a\vec{y}))$
 $= L(\vec{x}) + L(a\vec{y})$
 $= L(\vec{x}) + aL(\vec{y})$

So L satisfies the second condition as well.

② \Rightarrow ①: Assume $L(\vec{x} + a\vec{y}) = L(\vec{x}) + aL(\vec{y})$ for
all $\vec{x}, \vec{y} \in \mathbb{R}^n$ and all $a \in \mathbb{R}$. Now

$$L(\vec{x} + \vec{y}) = L(\vec{x} + 1 \cdot \vec{y}) = L(\vec{x}) + 1L(\vec{y}) = L(\vec{x}) + L(\vec{y}).$$

$$L(a\vec{x}) = L(\vec{0} + a\vec{x}) = L(\vec{0}) + aL(\vec{x})$$
$$= \vec{0} + aL(\vec{x}) = aL(\vec{x})$$

So L satisfies the first condition as well. \square

Ex: Let $M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. I claim $L: \mathbb{R}^2 \rightarrow \mathbb{R}^2$

defined by $L(\vec{x}) = M\vec{x}$ is a linear map.

Indeed, supposing $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ and $a \in \mathbb{R}$.

$$L(\vec{x} + \vec{y}) = M(\vec{x} + \vec{y}) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 + x_2 + y_2 \\ x_2 + y_2 \end{bmatrix}$$

$$\text{and } L(\vec{x}) + L(\vec{y}) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Same output!

$$= \begin{bmatrix} x_1 + x_2 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 + y_2 \\ y_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 + x_2 + y_1 + y_2 \\ x_2 + y_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 + x_2 + y_2 \\ x_2 + y_2 \end{bmatrix}$$

So we have $L(\vec{x} + \vec{y}) = L(\vec{x}) + L(\vec{y})$ in this case.

$$L(a\vec{x}) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \left(a \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} ax_1 \\ ax_2 \end{bmatrix}$$

$$= \begin{bmatrix} ax_1 + ax_2 \\ ax_2 \end{bmatrix} = \begin{bmatrix} a(x_1 + x_2) \\ ax_2 \end{bmatrix} = a \begin{bmatrix} x_1 + x_2 \\ x_2 \end{bmatrix} = aL(\vec{x})$$

Thus, L is a linear map! \square

Let M be an $m \times n$ matrix. Then M determines a linear map $L_M: \mathbb{R}^n \rightarrow \mathbb{R}^m$ via $L_M(\vec{x}) = M\vec{x}$.

Point: Matrices give linear maps 😊.

Ex: Let $M = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 3 \end{bmatrix}$. The associated linear map.

① has domain \mathbb{R}^3

② has codomain \mathbb{R}^2

$$L_M(\vec{x}) = M\vec{x} = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 + 2x_2 + x_3 \\ -x_1 + x_2 + 3x_3 \end{bmatrix} \leftarrow 2 \times 1$$

$$= \begin{bmatrix} x_1 \\ -x_1 \end{bmatrix} + \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} + \begin{bmatrix} x_3 \\ 3x_3 \end{bmatrix}$$

$$= x_1 \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}} + x_2 \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} + x_3 \underbrace{\begin{bmatrix} 1 \\ 3 \end{bmatrix}}$$

So in this example, L_M takes each vector \vec{x} to a linear combination of the columns of M ... This happens in General!

Prop: If $M = [\vec{c}_1 | \vec{c}_2 | \dots | \vec{c}_n]$ has columns $\vec{c}_1, \vec{c}_2, \dots, \vec{c}_n$, then the linear map $L_M: \mathbb{R}^n \rightarrow \mathbb{R}^m$ has formula

$$L_M \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \vec{c}_1 + x_2 \vec{c}_2 + \dots + x_n \vec{c}_n.$$

In particular, every range-value of L_M is a linear combination of the columns of M .

Ex: Write the range values of L_M as a linear combination of vectors for matrix

$$M = \begin{bmatrix} -1 & -2 & 1 \\ 3 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 3 & 0 \end{bmatrix}$$

Note $L_M: \mathbb{R}^3 \rightarrow \mathbb{R}^4$ as a function. Moreover

$$L_M \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} -1 \\ 3 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Hence range (L_M) = $\left\{ s \begin{bmatrix} -1 \\ 3 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 0 \\ 3 \end{bmatrix} + u \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} : s, t, u \in \mathbb{R} \right\}$

NB: the range of function $f: S \rightarrow T$ is

$$\text{range}(f) := \{ t : t = f(s) \text{ for some } s \in S \}$$

i.e. $\text{range}(f) = \{ f(s) : s \in \text{dom}(f) \}$.

NB: I keep saying "if L is determined by a matrix..."

Actually, every linear map is determined by a matrix...

→ Proof coming soon (but not too soon :-).

Back to linear systems: ^{represents}

If $[M|\vec{b}]$ ~~is~~ a linear system, then the solutions of the system satisfy $M\vec{x} = \vec{b}$. i.e.

$$L_M(\vec{x}) = M\vec{x} = \vec{b}, \text{ so } [M|\vec{b}] \text{ has a}$$

solution if and only if $\vec{b} \in \text{range}(L_M)$.

in other words, \vec{b} is a linear combination of the columns of M ... i.e. range elements of L_M correspond to solvable linear systems with matrix of coefficients M . 😊

Ex: Is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ in the range of $L(\vec{x}) = \begin{bmatrix} 3x - y + z \\ -x + y + z \end{bmatrix}$?

Sol: $L\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \iff \begin{bmatrix} 3x - y + z \\ -x + y + z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\iff \begin{bmatrix} 3 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\iff \left[\begin{array}{ccc|c} 3 & -1 & 1 & 2 \\ -1 & 1 & 1 & 1 \end{array} \right] \text{ has a solution...}$$

$$\left[\begin{array}{ccc|c} 3 & -1 & 1 & 2 \\ -1 & 1 & 1 & 1 \end{array} \right] \rightsquigarrow \left[\begin{array}{ccc|c} 0 & 2 & 4 & 5 \\ -1 & 1 & 1 & 1 \end{array} \right]$$

\rightsquigarrow [Finish for homework